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NAT... ...NICAL
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SCHOOL OF ENGINEERING
AND APPLIED SCIENCE

UNCLASSIFIED
Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Department of Civil, Mechanical, & Environmental Engineering, The George Washington University, Washington, D.C.	2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED
	2b. GROUP

3. REPORT TITLE

AN APPROXIMATE ANALYSIS OF THE BUCKLING OF IMPERFECT SPHERICAL SHELLS

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Technical Report, 1966-1971

5. AUTHOR(S) (First name, middle initial, last name)

Barry I. Hyman

6. REPORT DATE	7a. TOTAL NO OF PAGES	7b. NO OF REFS
January 1972	37	17

8a. CONTRACT OR GRANT NO N00014-66-C0154	9a. ORIGINATOR'S REPORT NUMBER(S) Technical Report No. 5
b. PROJECT NO Ship Systems Command	
c. Subproject SF 013 03 02	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
d. Task 1960	

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11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Structures Department Naval Ship Research and Development Center

13. ABSTRACT

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I

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Spherical Shells						
Snap Through Buckling						
Elastic Stability						

II

AN APPROXIMATE ANALYSIS
OF THE
BUCKLING OF IMPERFECT
SPHERICAL SHELLS

by

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This research was sponsored by the Structures Department of the Naval Ship Research and Development Center under Naval Ship Systems Command Subproject SF 013 03 02, Task 1960, Contract N00014-66-C-0154.

Technical Report No. 5

III

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ABSTRACT

An approximate method is presented for predicting elastic collapse of complete spherical shells subject to uniform external pressure. The shell contains an imperfection in the form of an isolated flat spot and the snap through behavior of the flat spot region is analyzed. The existence of higher modes is demonstrated and the effect of various choices for the stiffness coefficients at the edge of the flat spot is investigated.

*The work reported here was supported by the Naval Ship Research and Development Center through Contract N00014-66-C0154. Partial support was also furnished through NASA Grant NGR-09-010-053.

INTRODUCTION

The effect of small imperfections on the elastic buckling strength of complete spherical shells subject to uniform external pressure has been well demonstrated both theoretically and experimentally in recent years. Several general purpose computer programs have the capability of analyzing a shell of revolution with arbitrary meridian and one of these programs [1] has been used to calculate buckling pressures for spherical shells with isolated flat spots. The predicted results agreed well with the experimental results reported in [2]. An analysis which is valid for spherical shells with arbitrary axisymmetric imperfections is presented in [3], while a Rayleigh-Ritz analysis of the effect of an isolated flat spot is given in [4]. In addition, an assessment of "imperfection sensitivity" of spherical shells is provided in [5].

The extension of any of these approaches to include asymmetric imperfections, such as that associated with the interaction of neighboring flat spots, would encounter serious computational difficulties. An indication of the complexities that would be involved is provided by [6], in which a clamped shallow cap with an asymmetric imperfection is analyzed. An alternative approach to the development of numerical solutions to the exact equations of asymmetrical large deflection shell theory, would be a search for approximate solutions which might

provide an estimate of the collapse strength albeit at the expense of an accurate representation of details of the stress distribution. The first step in this search is described in this paper.

Another extension of the problem of the elastic collapse of the imperfect spherical shell would be the inclusion of local plastic effects in the flat spot region. The importance of local yielding on the collapse strength of imperfect spherical shells has been studied in [7] using the finite element method. However an approximate analysis which does not require extensive computer time would be a significant aid in practical design work.

The specific problem to be analyzed in this paper is the axisymmetric elastic collapse of externally pressurized complete spherical shells possessing a local flat spot. The flat spot region is modeled as a shallow cap which is elastically supported around its edge by the "remainder" of the shell. This is the same problem studied in [4] using the Rayleigh-Ritz method. The method to be described in this paper is an extension of that used in [8] for the analysis of clamped spherical caps and studied in more detail in [9]. It was shown in [9] that points on the load-deflection curve of the cap can be determined simply by solving a quadratic equation. It is felt that this method, because of its inherent simplicity, would be suitable for later generalization to either the problem of multiple, nonisolated,

flat spots or the elastic-plastic problem if it can be shown to yield a satisfactory solution to the isolated flat spot elastic problem.

In Ref. [9] it was shown that the snap through analysis of the clamped spherical cap exhibited higher mode solutions. It was postulated in [9], and confirmed independently in [10], that these higher mode solutions can be of significance in explaining the snap through behavior of the clamped cap, particularly when these higher modes merge with the lowest branch of the load deflection curve. The existence of higher mode solutions for the imperfect spherical shell problem will be investigated in this paper.

Also to be studied in this paper is the question of the choice of stiffness coefficients at the edge of the flat spot region. Since the flat spot region will be studied with the use of an approximate analysis, it is appropriate to also consider the use of approximate stiffness coefficients to represent the behavior of the "remainder" of the shell.

DERIVATION OF GOVERNING EQUATIONS

Shown in Fig. 1 is the spherical shell of nominal radius a and thickness h, with a local flat spot of radius $R_0 > a$ extending over a base circle of radius R. The shell is subject to uniform external pressure q. Diagrams of the flat spot region and the remainder of the sphere are shown in Fig. 2 which also defines the positive directions for the force and moment resultants, H_R and M_R, and the horizontal and vertical displacements, u and w, in the flat spot region.

It is assumed that elastic collapse of the shell is caused by snap through of the flat spot, and that the large deflections which accompany snap through are confined to the flat spot region identified in Fig. 2; then the deformation of the remainder can be characterized by linear theory. This is the same assumption made in [4]. Evidence of the validity of this assumption is provided in [1] and [3] where plots of the displacement as obtained by numerical solution of the exact nonlinear equations show that, even at the pressures corresponding to snap through collapse, the displacements of the remainder are within the range of linear theory.

The total potential energy can be expressed as

$$\Pi = \Pi_1 + \Pi_2 \tag{1}$$

where Π_1 represents the total potential energy of the flat spot, and Π_2 is the total potential energy of the remainder.

For symmetrical deformations of the flat spot, considered as a shallow spherical shell subject to uniform lateral pressure q, the total potential energy is

$$\Pi_1 = \frac{\pi E h^3}{12(1-\nu^2)} \int_0^R \{[\, e_1^2 - 2(1-\nu) e_2] \left(\frac{12}{h^2}\right) + (\nabla^2 w)^2$$

$$- \frac{2(1-\nu)}{r} \frac{dw}{dr} \frac{d^2 w}{dr^2} \} \, r dr - 2\pi q \int_0^R w r dr \qquad (2)$$

where

$$e_1 = \frac{du}{dr} + \frac{u}{r} + \frac{1}{2} \left(\frac{dw}{dr}\right)^2 + \frac{r}{R_0} \frac{dw}{dr} \qquad (3)$$

$$e_2 = \frac{u}{r} \frac{du}{dr} + \frac{u}{2r} \left(\frac{dw}{dr}\right)^2 + \frac{u}{R_0} \frac{dw}{dr} \qquad (4)$$

are respectively, the first and second invariants of the middle surface strains and

$$\nabla^2 w = \frac{d^2 w}{dr^2} + \frac{1}{r} \frac{dw}{dr} \qquad (5)$$

The approximate method to be used in this paper involves setting $e_2 = 0$ in equation (2). This assumption was used by Nash and Modeer in [8] for the analysis of clamped spherical caps, and was interpreted by Nash and Modeer as being equivalent to the assumption that the radial membrane force in the shell is constant. The snap through pressure of clamped caps, computed using this assumption, was shown in [9] to be within about 15% of the exact values as long as the geometric parameter $\lambda = 2[3(1-\nu^2)]^{1/4} R/(2R_0 h)^{1/2}$ was less than $\lambda = 5$. Since

Bushnell [11] has shown (see his Figure 6) that for $\lambda > 5$, the imperfect sphere will fail due to bifurcation of the flat spot region into the asymmetric mode, it is felt that setting $e_2 = 0$ for the elastically supported cap (flat spot region) will introduce no more than the 15% error obtained in the case of the clamped cap. In fact, the relaxation of the clamped conditions at the edge of the cap is likely to make the assumption of constant radial membrane force a more realistic approximation than it is for the clamped cap. Figure 5 in Reference [1] lends support to this argument.

Denoting the displacements at the intersection of the flat spot region with the remainder by u_R and w_R (see Figures 1 and 2), the total potential energy of the remainder can be written as [4]

$$\Pi_2 = 2\pi R \left[\frac{1}{2} M_R \left(\frac{dw}{dr}\right)_R - \frac{1}{2} H_R u_R \right] \tag{6}$$

If the following relations are used

$$\begin{aligned} H_R &= H_{R_1} + H_{R_2} \\ u_R &= u_{R_1} + u_{R_2} \end{aligned} \tag{7}$$

where the subscripts 1 refer to membrane quantities and the subscripts 2 denote the effects of bending, then equation (6) can be written as

$$\begin{aligned} \Pi_2 = 2\pi R \Big[& \frac{1}{2} M_R \left(\frac{dw}{dr}\right)_R - \frac{1}{2} H_{R_1} u_{R_1} - H_{R_1} u_{R_1} \\ & - \frac{1}{2} H_{R_2} u_{R_2} \Big] \end{aligned} \tag{8}$$

where the principle of reciprocity has been used to set

$$\frac{1}{2} H_{R_2} u_{R_1} = \frac{1}{2} H_{R_1} u_{R_2}$$

Assuming that linear shell theory is adequate for describing the behavior of the remainder, H_{R_2} and M_R can be expressed in terms of stiffness coefficients as

$$H_{R_2} = K_{11} u_{R_2} + K_{12} \left(\frac{dw}{dr}\right)_R$$

$$M_R = K_{21} u_{R_2} + K_{22} \left(\frac{dw}{dr}\right)_R \tag{9}$$

From membrane theory, we have

$$H_{R_1} = -\frac{qa\sqrt{1-(R/a)^2}}{2} \simeq -qa/2 \tag{10}$$

$$u_{R_1} = -qa(1-\nu)R/Eh \tag{11}$$

Utilizing equations (2), (9), (10) and (11) together with the assumption that $e_2 = 0$ allows equation (1) to be written as

$$\Pi = \frac{\pi Eh^3}{12(1-\nu^2)} \int_0^R \left[\frac{12e_1^2}{h^2} + (\nabla^2 w)^2 - \frac{2(1-\nu)}{r} \frac{dw}{dr} \frac{d^2w}{dr^2} \right] rdr$$

$$- 2\pi q \int_0^R wrdr + \pi R \left[K_{22} \left(\frac{dw}{dr}\right)_R^2 + (K_{21} - K_{12}) \left(\frac{dw}{dr}\right)_R u_{R_2}\right.$$

$$\left. - K_{11} (u_{R_2})^2 - \frac{q^2a^2(1-\nu)R}{2Eh} + qa\, u_{R_2} \right] \tag{12}$$

Setting the first variation of equation (12) with respect to u and w equal to zero yields the two governing differential equations [8]

$$e_1 = -\beta^2h^2/12 \tag{13}$$

$$\nabla^4 w + \beta^2\nabla^2 w = \frac{q}{D} - \frac{2\beta^2}{R_0} \tag{14}$$

where β is a constant of integration. The associated conditions at $r = R$ are

$$w_R = 0 \tag{15}$$

$$e_1 + \frac{(1-\nu^2)}{2Eh}\left[(K_{21} - K_{12})\left(\frac{dw}{dr}\right)_R - K_{11}\,u_{R_2} + qa\right] = 0 \tag{16}$$

$$R\left(\frac{d^2w}{dr^2}\right)_R + \nu\left(\frac{dw}{dr}\right)_R + \frac{R}{D}\left[K_{22}\left(\frac{dw}{dr}\right)_R + \frac{1}{2}(K_{21} - K_{12})\,u_{R_2}\right] = 0 \tag{17}$$

where

$$D = Eh^3/12(1-\nu^2).$$

SOLUTION

The general solution to equation (14) is [8]

$$w = C_1J_0(\beta r) + C_2 + \frac{r^2}{4\beta^2}\left(\frac{q}{D} - \frac{2\beta^2}{R_0}\right) \tag{18}$$

where those solutions which are unbounded at $r = 0$ have been suppressed and C_1 and C_2 are constants of integration. Then u is determined from equations (3), (13) and (18) as

$$u = \frac{-\beta^2 h^2 r}{24} - \frac{1}{2r}\{C_1{}^2 \frac{\beta^2 r^2}{2}[J_1{}^2(\beta r) - J_0(\beta r)J_2(\beta r)]$$

$$- \frac{C_1 q}{\beta^2 D} r^2 J_2(\beta r) + \frac{r^4}{16\beta^4}(\frac{q}{D} - \frac{2\beta^2}{R_0})^2\} + \frac{r^3}{8\beta^3 R_0}(\frac{q}{D} - \frac{2\beta^2}{R_0}) \quad (19)$$

J_0, J_1, J_2 appearing in the above equations are the Bessel functions of zeroth, first and second order. It is convenient at this time to non-dimensionalize the equations by introducing the following parameters

$$\rho = r/R$$

$$\omega = w/h$$

$$Q = \frac{\sqrt{3(1-\nu^2)}a^2 q}{2Eh^2}$$

$$x = \beta R$$

$$\theta^4 = 12(1-\nu^2)R^4/a^2h^2 \qquad [\text{Note: } \lambda^4 = \theta^4 (\frac{a}{R_0})^2]$$

$$\mu = uR/h^2$$

$$K_{11}^* = \frac{\theta^2}{E[12(1-\nu^2)]^{1/4}}(\frac{a}{h})^{1/2} K_{11}$$

$$K_{22}^* = \frac{[12(1-\nu^2)]^{3/4}}{Eh^2}(\frac{a}{h})^{1/2} K_{22}$$

$$K_{12}^* = \frac{[12(1-\nu^2)]^{1/4}\theta}{Eh}(\frac{a}{h})^{1/2} K_{12}$$

$$K_{21}^* = \frac{[12(1-\nu^2)]^{1/4}\theta}{Eh}(\frac{a}{h})^{1/2} K_{21}$$

$$C_1^* = C_1/h$$

$$C_2^* = C_2/h \quad (20)$$

Then equations (18) and (19) transform to

$$\omega = C_1^* J_o(x\rho) + C_2^* + \frac{\rho^2\theta^2}{2[12(1-\nu^2)]^{1/2}} [\frac{2\theta^2 Q}{x^2} - (\frac{a}{R_0})] \tag{21}$$

$$\mu = \frac{\rho}{2} \{- \frac{x^2}{12} - C_1^{*2} \frac{x^2}{2} [J_1{}^2(x\rho) - J_o(x\rho) J_2(x\rho)] + \frac{4\theta^4 Q\, J_2(x\rho) C_1^*}{x^2[12(1-\nu^2)]^{1/2}} + \frac{\theta^4\rho^2}{48(1-\nu^2)} [(\frac{a}{R_o})^2 - \frac{4\theta^4 Q^2}{x^4}]\} \tag{22}$$

The boundary conditions (15) - (17) become, in nondimensionalized form

$$\omega|_{\rho=1} = 0 \tag{23}$$

$$\{[12(1-\nu^2)]^{1/2} (K_{21}^* - K_{12}^*) \frac{d\omega}{d\rho} - \frac{12(1-\nu^2)}{\theta} K_{11}^*\mu\}_{\rho=1} = x^2 - 4Q\theta^2 \tag{24}$$

$$\{\frac{d^2\omega}{d\rho^2} + \nu \frac{d\omega}{d\rho} + \theta K_{22}^* \frac{d\omega}{d\rho} + \frac{[12(1-\nu^2)]^{1/2}}{2} (K_{21}^* - K_{12}^*)\mu_2\}_{\rho=1} = 0 \tag{25}$$

The imposition of the first of these boundary conditions leads directly to the following expression for C_2^*

$$C_2^* = - C_1^* J_o(x) - \frac{\theta^2}{2[12(1-\nu^2)]^{1/2}} [\frac{2\theta^2 Q}{x^2} - (\frac{a}{R_o})] \tag{26}$$

Substituting equations (21), (22) and the nondimensionalized form of (11) into the remaining two boundary conditions (24) and (25) results in

$$B_1 (C_1^*)^2 + (B_2 + B_3 Q)\ C_1^* + D_1 + D_2 Q + D_3 Q^2 = 0 \tag{27}$$

$$B_4 (C_1^*)^2 + (B_5 + B_6 Q)\ C_1^* + D_4 + D_5 Q + D_6 Q^2 = 0 \tag{28}$$

where the $B_1 \ldots B_6$, $D_1 \ldots D_6$ coefficients do not explicitly involve the pressure parameter Q and are listed below:

$$B_1 = \frac{[12(1-\nu^2)]^{3/2}}{48\theta^3} \left[J_1^{\,2}(x) - J_o(x) J_2(x)\right] x^2 K_{11}^*$$

$$B_2 = \frac{2(1-\nu^2) x J_1(x) K_{12}^*}{\theta^2}$$

$$B_3 = \frac{-2(1-\nu^2)\theta}{x^2} J_2(x)\ K_{11}^*$$

$$B_4 = \frac{[12(1-\nu^2)]^{3/4}}{4\theta} \left[J_1^{\,2}(x) - J_o(x) J_2(x)\right] x^2 K_{12}^*$$

$$B_5 = \frac{[12(1-\nu^2)]^{1/4}}{\theta} \left[(1-\nu) J_1(x) - x J_o(x) - \theta J_1(x) K_{22}^*\right] x$$

$$B_6 = -2[12(1-\nu^2)]^{1/4}\ \frac{\theta^3 J_2(x) K_{12}^*}{x^2}$$

$$D_1 = \frac{[12(1-\nu^2)]^{1/2}}{12} \left\{ \frac{-x^2}{\theta^2} - \frac{(1-\nu^2) K_{11}^*}{2\theta^3} \left[\frac{\theta^4}{4(1-\nu^2)} \left(\frac{a}{R_o}\right)^2 - x^2 \right] + 2\left(\frac{a}{R_o}\right) K_{12}^* \right\}$$

$$D_2 = \frac{[12(1-\nu^2)]^{1/2}}{6} \left[2 - \frac{(1-\nu)}{\theta} K_{11}^* - 2 \frac{\theta^2}{x^2} K_{12}^* \right]$$

$$D_3 = \frac{[12(1-\nu^2)]^{1/2}}{24x^4} \theta^5 K_{11}^*$$

$$D_4 = \frac{-1}{[12(1-\nu^2)]^{1/4}} \left\{ \left(\frac{a}{R_o}\right) \theta (1+\nu) + \frac{(1-\nu^2)}{2\theta} K_{12}^* \left[\frac{\theta^4}{4(1-\nu^2)} \left(\frac{a}{R_o}\right)^2 - x^2 \right] \right.$$

$$\left. + \theta^2 \left(\frac{a}{R_o}\right) K_{22}^* \right\}$$

$$D_5 = \frac{2\theta}{[12(1-\nu^2)]^{1/4}} \left[\frac{\theta^2 (1+\nu)}{x^2} - (1-\nu) K_{12}^* + \frac{\theta^3}{x^2} K_{22}^* \right]$$

$$D_6 = \frac{\theta^7 K_{12}^*}{2x^4 [12(1-\nu^2)]^{1/4}} \tag{29}$$

The reciprocity theorem was used to set $K_{21}^* = -K_{12}^*$ in the derivation of the above expressions.

If equation (27) is multiplied by B_4, equation (28) by B_1, and then the two resulting expressions subtracted from each other, the following equation for C_1^* is obtained:

$$C_1^* = \frac{B_1(D_4 + D_5 Q + D_6 Q^2) - B_4(D_1 + D_2 Q + D_3 Q^2)}{(B_4 B_2 - B_1 B_5) + (B_4 B_3 - B_1 B_6) Q} \tag{30}$$

This is then substituted back into equation (27) to obtain, after some lengthy algebra,

$$G_4 Q^4 + G_3 Q^3 + G_2 Q^2 + G_1 Q + G_0 = 0 \tag{31}$$

where

$$G_4 = (B_1D_6-B_4D_3)^2B_1 + (B_1D_6-B_4D_3)(B_4B_3-B_1B_6)B_3$$

$$+ (B_4B_3-B_1B_6)^2D_3$$

$$G_3 = 2\ (B_1D_5-B_4D_2)(B_1D_6-B_4D_3)\ B_1$$

$$+ (B_1D_5-B_4D_2)(B_4B_3-B_1B_6)\ B_3$$

$$+ (B_1D_6-B_4D_3)\ [2B_2B_3B_4-B_1(B_2B_6+B_3B_5)]$$

$$+ D_2(B_4B_3-B_1B_6)^2 + 2D_3(B_4B_2-B_1B_5)(B_4B_3-B_1B_6)$$

$$G_2 = 2B_1(B_1D_4-B_4D_1)(B_1D_6-B_4D_3) + (B_1D_5-B_4D_2)^2B_1$$

$$+ (B_1D_4-B_4D_1)(B_4B_3-B_1B_6)B_3$$

$$+ (B_1D_5-B_4D_2)\ [2B_2B_3B_4-B_1(B_2B_6+B_3B_5)]$$

$$+ (B_1D_6-B_4D_3)(B_4B_2-B_1B_5)B_2$$

$$+ D_1(B_4B_3-B_1B_6)^2 + 2D_2(B_4B_2-B_1B_5)(B_4B_3-B_1B_6)$$

$$+ D_3(B_4B_2-B_1B_5)^2$$

$$G_1 = 2B_1(B_1D_4-B_4D_1)(B_1D_5-B_4D_2)$$

$$+ (B_1D_4-B_4D_1)\ [2B_2B_3B_4-B_1(B_2B_6+B_3B_5)]$$

$$+ B_2(B_1D_5-B_4D_2)\ (B_4B_2-B_1B_5)$$

$$+ 2D_1(B_4B_2-B_1B_5)\ (B_4B_3-B_1B_6) + D_2(B_4B_2-B_1B_5)$$

$$G_0 = (B_1D_4-B_4D_1)^2\ B_1 + (B_1D_4-B_4D_1)\ (B_4B_2-B_1B_5)B_2$$

$$+ D_1(B_4B_2-B_1B_5)^2 \tag{32}$$

Hence, the equation for determining Q is a quartic, as compared to a quadratic equation for Q in the case of the clamped cap. However, by utilizing equations (29) it is found that

$$B_1D_6 - B_4D_3 \equiv 0$$

$$B_4B_3 - B_1B_6 \equiv 0 \tag{33}$$

This leads to $G_4 = G_3 = 0$ so that equation (31) does in fact simplify to a quadratic in Q

$$G_2Q^2 + G_1Q + G_0 = 0 \tag{34}$$

with

$$G_2 = T_1{}^2B_1 + T_1T_2B_3 + T_2{}^2D_3$$

$$G_1 = 2B_1T_1T_3 + T_3T_2B_3 + T_2{}^2D_2 + B_2T_1T_2$$

$$G_0 = T_3{}^2B_1 + T_2T_3B_2 + T_2{}^2D_1 \tag{35}$$

where

$$T_1 = B_1D_5 - B_4D_2$$

$$T_2 = B_4B_2 - B_1B_5$$

$$T_3 = B_1D_4 - B_4D_1 \tag{36}$$

Since the G_2, G_1, G_0 coefficients are homogeneous in T_1, T_2, T_3, any factor common to T_1, T_2, T_3 can be cancelled. Substituting from equations (29) into equations (36) yields

$$T_1 = K_{11}^* \theta(1+\nu) - 2K_{12}^* x^2 + \theta^2(K_{11}^* K_{22}^* + 2K_{12}^{*\,2})$$

$$T_2 = \frac{[12(1-\nu^2)]^{1/2}}{2\theta^2} x^3 \{J_1(x)(K_{11}^* K_{22}^* + 2K_{12}^{*\,2})$$

$$-\frac{K_{11}^*}{\theta} [(1-\nu)J_1(x) - x J_0(x)]\}$$

$$T_3 = \frac{[12(1-\nu^2)]^{1/2}}{2\theta^2} x^2 [x^2 K_{12}^* - (\frac{a}{R_0}) \theta(1+\nu) K_{11}^*$$

$$- \theta^2 (\frac{a}{R_0})(K_{11}^* K_{22}^* + 2K_{12}^{*\,2})] \tag{37}$$

where the common factor $[12(1-\nu^2)]^{5/4}$ $[J_1^2(x) - J_0(x)J_2(x)]/24\theta$ has been cancelled.

The load-deflection curve for a given shell $(\theta, a/R_0, \nu)$ is obtained as follows: For a given value of the strain parameter x, the coefficients $B_1, B_2, B_3, D_1, D_2, D_3, T_1, T_2, T_3$ are evaluated using equations (29) and (37). Then the coefficients G_2, G_1, G_0 are found from equations (35). From equation (34), two values of the pressure Q are found. For each value of pressure, the constants C_1^* and C_2^* are determined from equations (26) and (30). Note that as a consequence of the relations (33), equation (30) reduces to

$$C_1^* = \frac{T_1 Q + T_3}{T_2} \tag{38}$$

With C_1^* and C_2^* determined, the displacements ω and μ are given as functions of position ρ by equations (21) and (22).

In order to generate the load-deflection curve it is convenient to introduce the average deflection parameter

$$\bar{\omega} = \frac{2[12(1-\nu^2)]^{1/2}}{\theta^2} \int_0^1 \omega\rho d\rho \tag{39}$$

Though $\bar{\omega}$ does not include the deflections of the "remainder" of the shell, it is still a suitable parameter for locating Q_{cr}, the first local maximum on the Q vs. $\bar{\omega}$ curve. Utilizing equations (21) and (26), $\bar{\omega}$ can be expressed as

$$\bar{\omega} = \frac{[12(1-\nu^2)]^{1/2}}{\theta^2} C_1^* J_2(x) - \frac{1}{4}\left[\frac{2\theta^2 Q}{x^2} - \left(\frac{a}{R_0}\right)\right] \tag{40}$$

Thus, for each value of x, the two values of Q and their associated values of $\bar{\omega}$ determine two points on the load-deflection curve. The lowest branch of the load deflection curve is generated by letting x increase from zero to the value x_M which makes $G_1^2 - 4G_2G_0$ vanish. Those values of $x > x_M$ for which $G_1^2 - 4G_2G_0$ is positive define points on the higher mode loops (see Ref. [9]).

As in [9] it is possible to obtain relatively simple expressions for the "fully snapped through" state (the configuration after snap through for which there is no middle surface strain) by considering the limiting values of Q and $\bar{\omega}$ as $x \to 0$. The roots of equation (34) are

$$Q_{1,2} = [-G_1 \pm (G_1^2 - 4G_2G_0)^{1/2}]/2G_2 \tag{41}$$

Combining equations (29), (37), and (35), it is seen that for $x << 1$, G_0 is of the order x^8 while both G_1 and G_2 are of the order of x^6. Thus

$$\lim_{x\to 0} Q_1 = 0$$

$$\lim_{x\to 0} Q_2 = \lim_{x\to o} \left(\frac{-G_1}{G_2}\right) \tag{42}$$

Evaluation of the limiting value of the ratio in the above equation leads to

$$\lim_{x\to 0} Q_2 = \frac{-96}{5K_{11}^{*}\theta^2} \tag{43}$$

It can also be shown that for $x << 1$, equation (40) reduces to

$$\bar{\omega} = \frac{\theta^2}{32}\left\{\frac{[K_{11}^{*}(3+\nu)+\theta(K_{11}^{*}K_{22}^{*}+2K_{12}^{*\,2})-32K_{12}^{*}/\theta]}{[K_{11}^{*}(1+\nu)+\theta(K_{11}^{*}K_{22}^{*}+2K_{12}^{*2})]} - \frac{4}{3}\right\} Q$$

so that utilizing the above expressions for the limiting values of Q, it is seen that

$$\lim_{x\to 0} \bar{\omega}_1 = 0$$

$$\lim_{x\to 0} \bar{\omega}_2 = \frac{-3}{5K_{11}^{*}}\left\{\frac{[K_{11}^{*}(3+\nu)+\theta(K_{11}^{*}K_{22}^{*}+2K_{12}^{*2})-32K_{12}^{*}/\theta]}{[K_{11}^{*}(1+\nu)+\theta(K_{11}^{*}K_{22}^{*}+2K_{12}^{*2})]} - \frac{4}{3}\right\} \tag{44}$$

The points $(Q_1,\bar{\omega}_1)$ and $(Q_2,\bar{\omega}_2)$ for $x = 0$ represent respectively the no load and "fully snapped through" states.

STIFFNESS COEFFICIENTS

The solution of the problem cannot be obtained without knowing the stiffness coefficients K_{11}^*, K_{12}^*, K_{22}^*. Essentially exact expressions for the stiffness coefficients of externally pressurized spherical shells including the pressure effect were derived by Bushnell [12]. Bushnell's expressions for the stiffness coefficients of the "remainder" of a spherical shell were simplified by Koga and Hoff [4] in their flat spot analysis for the case of an almost complete "remainder." This corresponds to assuming that the flat spot is small in extent ($R/a \ll 1$), an assumption which has already been used in equation (10) of this paper. The stiffness coefficients used in [9] are:

$$K_{11}^* = \frac{\Delta^*}{\Delta_o^*} (1-Q^2)^{1/2} \; (R_1{}^2 + I_1{}^2)$$

$$K_{12}^* = \frac{\Delta^*}{\Delta_o^*} \left[\left(\frac{1+Q}{2}\right)^{1/2} (R_0 I_1 - I_0 R_1) - \left(\frac{1-Q}{2}\right)^{1/2} (I_0 I_1 + R_0 R_1) \right]$$

$$\begin{aligned} K_{22}^* = -\frac{\Delta^*}{\Delta_o^*} \{ & (1-Q^2)^{1/2} \; (R_0{}^2 + I_0{}^2) \\ & + (1-\nu^2)(1-Q^2)^{1/2} \; (R_1{}^2 + I_1{}^2)/\theta^2 \\ & + 2[1-Q(1+\nu)] \left(\frac{1+Q}{2}\right)^{1/2} (R_1 I_0 - R_0 I_1)/\theta \\ & - 2[1+Q(1+\nu)] \left(\frac{1-Q}{2}\right)^{1/2} (R_0 R_1 + I_0 I_1)/\theta \; \} \end{aligned} \tag{45}$$

where

$$\Delta^* = (1+2Q)\left(\frac{1-Q}{2}\right)^{1/2}(R_1R_0 + I_1I_0) + (1-2Q)\left(\frac{1+Q}{2}\right)^{1/2}(I_1R_0 - R_1I_0)$$

$$+ (\nu-1)(1-Q^2)^{1/2}\quad (R_1^{\,2} + I_1^{\,2})/\theta$$

$$\Delta_0^* = (1-\nu^2)(1-Q^2)\ (R_1^{\,2} + I_1^{\,2})^2/\theta^2$$

$$+ 2[1-Q(1+\nu)](1+Q)\left(\frac{1-Q}{2}\right)^{1/2}\quad (R_1I_0 - R_0I_1)(R_1^{\,2} + I_1^{\,2})/\theta$$

$$- 2[1+Q(1+\nu)](1-Q)\left(\frac{1+Q}{2}\right)^{1/2}\quad (R_1R_0 + I_0I_1)(R_1^{\,2} + I_1^{\,2})/\theta$$

$$+ (1+2Q)(1-Q)[(R_0R_1)^2 + (I_0I_1^{\,2})]/2$$

$$+ (1-2Q)(1+Q)[(I_0R_1)^2 + (R_0I_1)^2]/2$$

$$+ 2QR_0I_0R_1I_1 - (1-Q^2)^{1/2}\quad [R_0I_0\ (R_1^{\,2} - I_1^{\,2})$$

$$+ R_1I_1\ (I_0^{\,2} - R_0^{\,2})] \tag{46}$$

R_0 and I_0 in the above equations are the real and imaginary parts respectively of the Hankel function of the first kind of order zero. Similarly R_1 and I_1 are the real and imaginary parts of the Hankel function of order one. The argument of the Hankel function is

$$z = \theta[Q + i\ (1- Q^2)]^{1/2} \tag{47}$$

If equations (45) are used in the expressions for $B_1, \ldots, T_3$ [equations (29) and (37)], it is readily apparent that the coefficients G_2, G_1, G_0 in equation (35) will involve the load Q

in an extremely complicated fashion. However the very simple explicit involvement of Q in equation (34) can still be used to advantage in obtaining a solution as follows: First, $B_1, \ldots T_3$ are evaluated by setting $Q = 0$ in equations (45)-(47). Then when a value of Q is determined from equation (34) this value is used to re-evaluate the stiffness coefficients from equations (45)-(47). This iterative procedure is continued until successive values of Q agree. Then the solution procedure continues with the determination of $\overline{\omega}$. There is of course no guarantee that the iteration process described above will converge. Iterative solutions for clamped caps do encounter convergence difficulties in the neighborhood of Q_{cr}, and in fact the pressure at which this loss of convergence occurs is often defined as Q_{cr} (see Ref. [9] for a more detailed discussion of this.) On the other hand, the advantage of extending the technique of [9] to the subject problem of this paper is that the complete load-deflection curve (including higher modes) can be generated without recourse to iterative solutions and their associated convergence difficulties. If the iteration process described above does lead to convergence difficulties, the drastic simplifications due to the use of the approximate solution for the flat spot region are negated. Hence it is appropriate to consider using less exact stiffness coefficients and in particular, stiffness coefficients which allow the load-deflection curve to be generated without resorting to iterative techniques.

The reason that the use of equations (45) for the stiffness coefficients requires iteration to generate the load-deflection curve is that the coefficients in equation (34) depend upon the loading parameter Q. If stiffness coefficients which are independent of Q are used instead of equations (45), then the coefficients in equation (34) will be independent of Q, and a closed form solution for Q is immediately obtainable. One such set of stiffness coefficients is obtained by setting $Q = 0$ in equations (45)-(47). The resulting stiffness coefficients are the shallow shell approximation to the stiffness coefficients obtained by Baker and Cline [13] in terms of Thomson functions. Since the effect of the external pressure on the stiffness of the "remainder" is ignored in these expressions, the results will become inaccurate if the loading on the complete sphere approaches the buckling load of the "remainder," i.e., as $Q \to 1$. Thus, the use of the Baker and Cline stiffness coefficients will not permit the present analysis to reduce to that of a perfect sphere as $(\frac{a}{R_0}) \to 1$. On the other hand, the analysis will become more accurate as $(\frac{a}{R_0})$ departs further from unity, i.e., as the flat spot becomes "flatter." Such an analysis serves to complement the imperfection sensitivity study of Hutchinson [5] which loses accuracy as $\frac{a}{R_0}$ departs from unity.

As indicated above, the involvement of Q in equations (45)-(47), not only explicitly but also in the argument of the Hankel functions, leads to concern regarding convergence

of any iteration process. This led to consideration of a set of stiffness coefficients which are independent of Q. Another approach to simplifying the problem while still retaining the effect of Q on the stiffness coefficients is to obtain asymptotic values for the stiffness coefficients for large values of θ. By taking the limit as $\theta \to \infty$, equations (45) reduce to

$$K^*_{11} = - [2(1-Q)]^{1/2}$$

$$K^*_{12} = 1$$

$$K^*_{22} = [2(1-Q)]^{1/2} \qquad (48)$$

These stiffness coefficients are the counterpart of those which were originally derived by Nachbar [14] for internally pressurized spherical shells. Nachbar's original derivation was based on the assumption that the edge angle of the "remainder" is close to $\Pi/2$, i.e., that the "remainder" is very nearly hemispherical. However no such restriction is required to derive equations (48) from equations (45), and since equations (45) are valid for an almost complete "remainder" (R/a << 1), so are equations (48). This raises an interesting point regarding the interpretation of the parameter

$$\theta^4 = 12(1-\nu^2)\left(\frac{R}{a}\right)^2 \left(\frac{R}{h}\right)^2$$

In most previous work $(\frac{R}{h})$ is considered as fixed, so increasing θ is associated with increasing $(\frac{R}{a})$ and hence a less shallow "remainder." However it is clear that $(\frac{R}{a})$ can be kept fixed and

increasing θ can then be interpreted as corresponding to a thinner "remainder." This latter interpretation, which is used in this paper, makes it possible to utilize equations (48) for other than nearly hemispherical "remainders."

Since equations (48) were derived by taking the limit as $\theta \to \infty$, the validity of using these expressions for the subject problem where the range of interest is $\theta < 6$ has to be examined. It was shown by Cline [15] in his study of the effect of internal pressure on the behavior of spherical shells that the influence coefficients rapidly approach their asymptotic values at relatively small values of θ. In fact Cline proposed that the asymptotic values of the influence coefficients be used for $\theta \geq 3$. For externally pressurized spherical shells, Bushnell [12] showed that when $Q > 0.5$ the influence coefficients become infinite at values of θ which depend on Q. However, according to Bushnell, the stiffness coefficients for externally pressurized spherical shells are well behaved and non-zero for all θ. It is reasonable then to assume that the stiffness coefficients for externally pressurized spherical shells possess asymptotic behavior as $\theta \to \infty$ which is similar to that of influence coefficients for internally pressurized spherical shells, i.e., the asymptotic form given by equations (48) are sufficiently accurate when $\theta \geq 3$.

The use of equations (48) instead of equations (45) represents a drastic simplification in the form of the coefficients

in equation (34). However these coefficients will still involve terms in which Q is raised to a non-integer power and a closed form solution for Q will not be obtainable. While the iteration procedure required to solve the governing equation is likely to be more stable than when equations (45) are used, there still cannot be any guarantee of convergence.

The asymptotic stiffness coefficients can be simplified even further by neglecting the pressure effect (setting $Q = 0$ in equations (48)). The expressions then reduce to the well known Geckeler [16] form. However whereas the Geckeler stiffness coefficients have previously been thought to be restricted to nearly hemispherical "remainders," the systematic derivation of these expressions from equations (45) shows that they are applicable to any sufficiently thin "remainder." The use of the Geckeler stiffness coefficients allows equation (45) to be solved directly via the quadratic formula. The use of these coefficients can be expected to lead to inaccuracies when either $\theta < 3$ or $\frac{a}{R_0} \approx 1$.

RESULTS

A computer program was written to evaluate the coefficients G_2, G_1, G_0 of equation (34) and to solve equation (34) either directly when G_2, G_1, G_0 are independent of Q or by iteration when G_2, G_1, G_0 involve Q. Solutions were obtained for the four sets of stiffness coefficients described in the previous section.

As expected, the use of either equations (45) or (48) did lead to convergence difficulties, primarily when attempting to compute values of Q in the neighborhood of unity. One reason for this can best be seen from an examination of equations (48) which become imaginary when $Q > 1$. Hence any iterative process which provides intermediate values for Q which are greater than unity is doomed to failure. Since one of the main purposes of undertaking the analysis described in this paper was to avoid convergence difficulties associated with iterative solutions of the problem, no attempt was made to refine the iteration process described earlier in this paper.

The computations revealed the presence of higher mode solutions and a load-deflection curve for $\theta = 6$, $a/R_0 = 1/1.05$ which includes a higher mode in the form of an isolated loop is shown in Figure 3. This result was obtained using the exact stiffness coefficients [equations (45)] and no convergence difficulties was encountered. Calculations using equation (48)

were also conducted for this case and the results for the main branch of the curve duplicate those shown in Figure 3. No higher mode solutions were sought with the use of equations (48). A systematic study of higher mode solutions for other values of θ, a/R_0 and other choices of the stiffness coefficients was not undertaken, since from the results shown in Figure 3 these solutions appear to be quite similar qualitatively to the higher mode solutions for clamped caps [9]. The influence of these higher modes on the snap-through behavior of the imperfect spherical shell can only be explored by refining the approximate solution for the behavior of the flat spot region along the lines described in [17].

A comparison of the load-deflection curves for various choices of the stiffness coefficients is shown in Figure 4 for $\theta = 3$, $a/R_0 = 1/1.15$. Notice that the effect of neglecting Q in the stiffness coefficients raises the value of Q_{cr}. This is to be expected since the effect of the external pressure on the "remainder" is to decrease its stiffness. It is interesting to note that the asymptotic stiffness coefficients [equations (48)] yield results which are in excellent agreement with those obtained using the exact stiffness coefficients, even at this small value of θ.

Curves similar to those of Figure 4 were generated for various values of θ and a/R_0. The snap through pressures obtained from these curves are plotted in Figure 5 along with the numerical results from [11]. It is seen that the approximate solutions

of this paper give relatively good results when $\theta \approx 3$. The errors increase as θ increases, particularly for the "flatter" imperfections. The results show that no significant advantage in accuracy can be gained by using more accurate stiffness coefficients for the remainder.

Several errors in the original version of equations (29) were found as this report was being prepared. They were corrected in the manuscript so equations (29) are correct as presented in this report. However these errors were discovered too late to be corrected in the computer program. Hence the results shown in Figures 3-5 are not correct. It is expected that the qualitative nature of the results shown in Figures 3 and 4 will not be affected by new computations based on the corrected equations; and that significant improvement in the accuracy of the results shown in Figure 5 will be achieved as a consequence of incorporating the corrections in the computer. These new results will be incorporated into the report prior to its release and distribution according to the attached list.

Since the results indicate that the use of the simplest stiffness coefficients (equations (48) with Q = 0) is appropriate for this problem, explicit evaluation of the fully snapped through configuration is possible. Substituting equations (48) with Q = 0 into equations (43) and (44) leads to

$$\lim_{x \to 0} Q_2 = \frac{6}{5\sqrt{2}} \left(\frac{4}{\theta}\right)^2$$

$$\lim_{x \to 0} \bar{\omega} = \frac{1}{5\sqrt{2}(1+\nu)} \left(5+2\nu+ \frac{96}{\sqrt{2}\theta}\right)$$

It is of interest to note that these values are independent of a/R_0 and that the value for Q_2 is almost identical to that obtained in [9] for the clamped cap.

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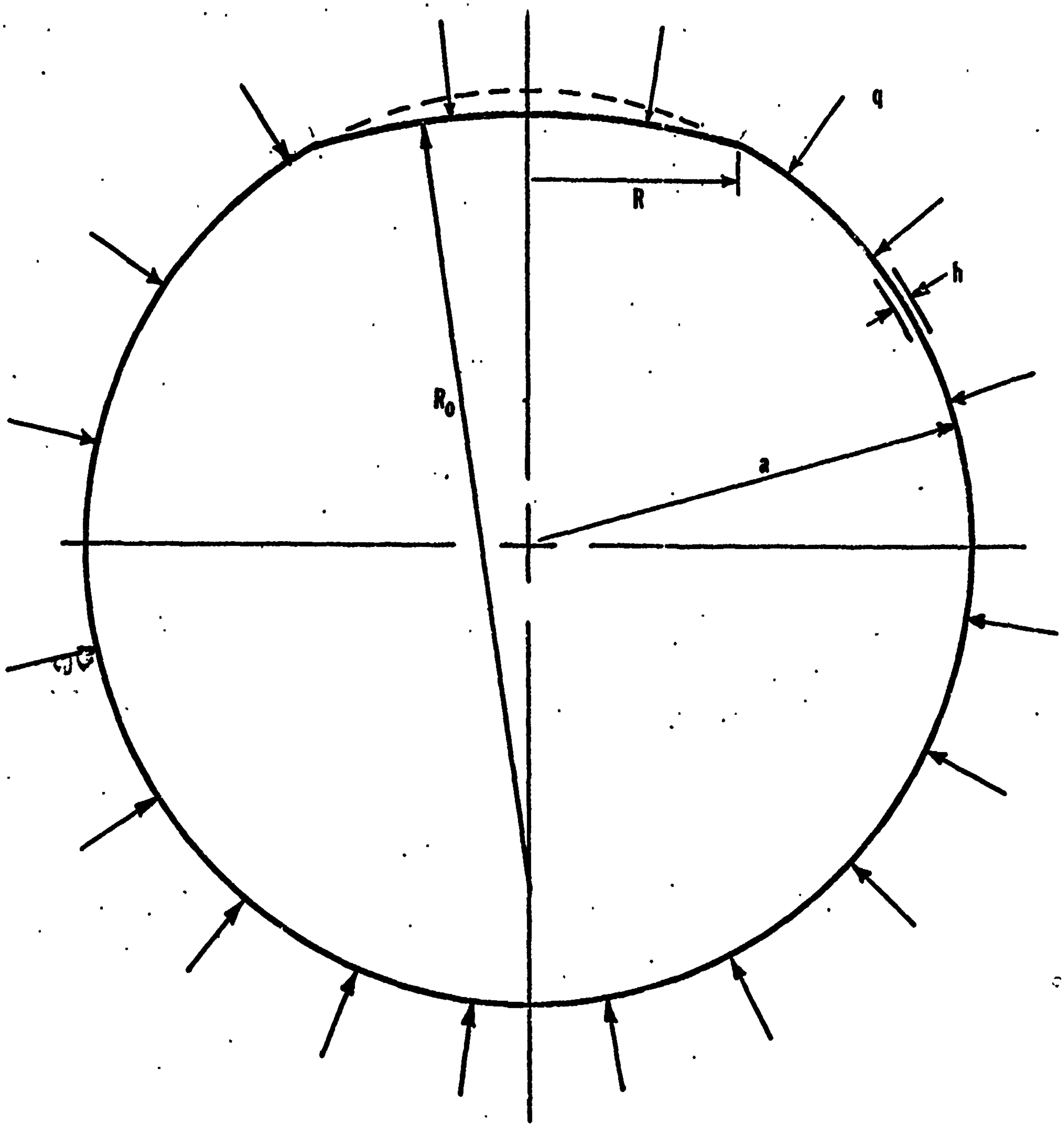

Figure 1. Geometry of Imperfect Spherical Shell

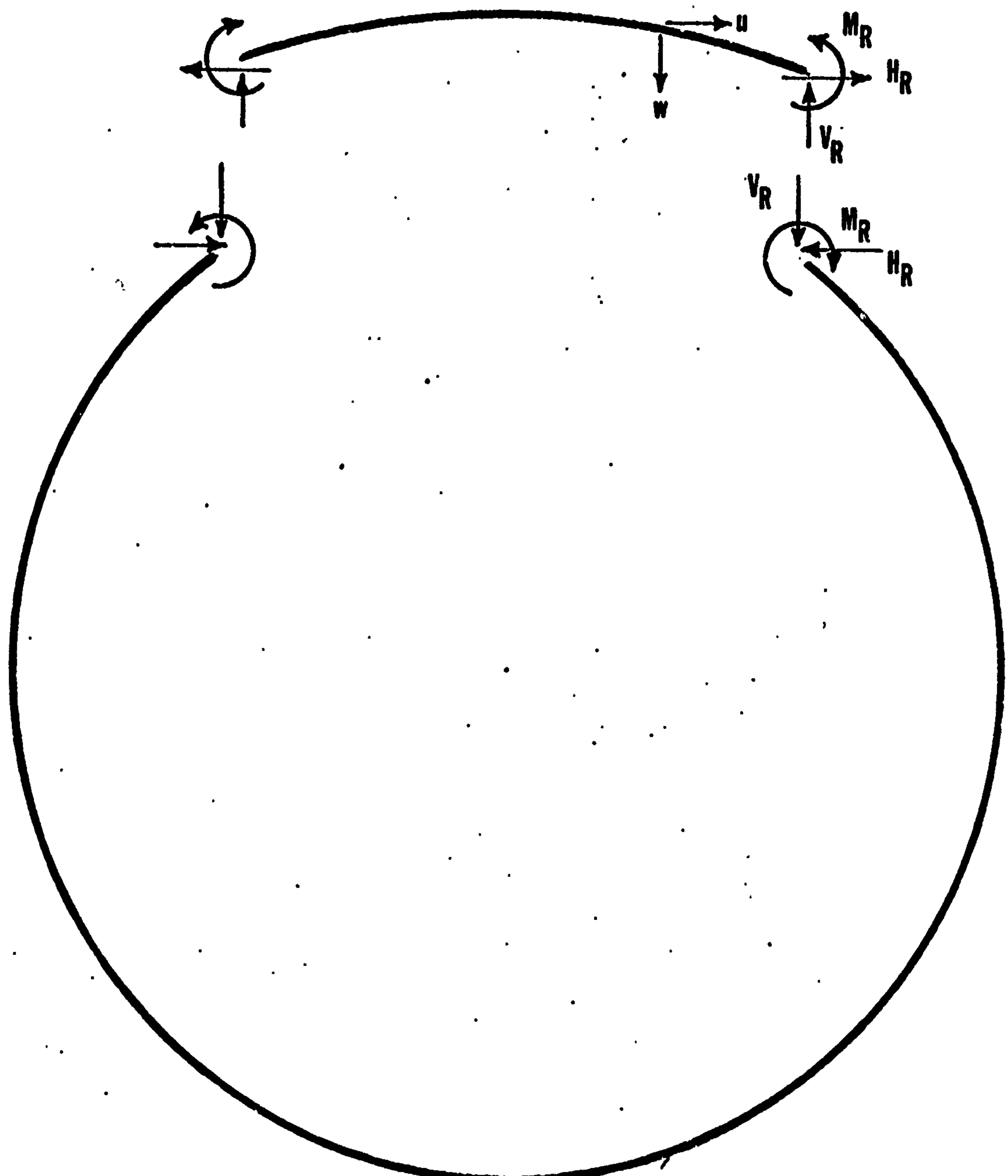

Figure 2. Force and Moment Resultants at Edge of Flat Spot

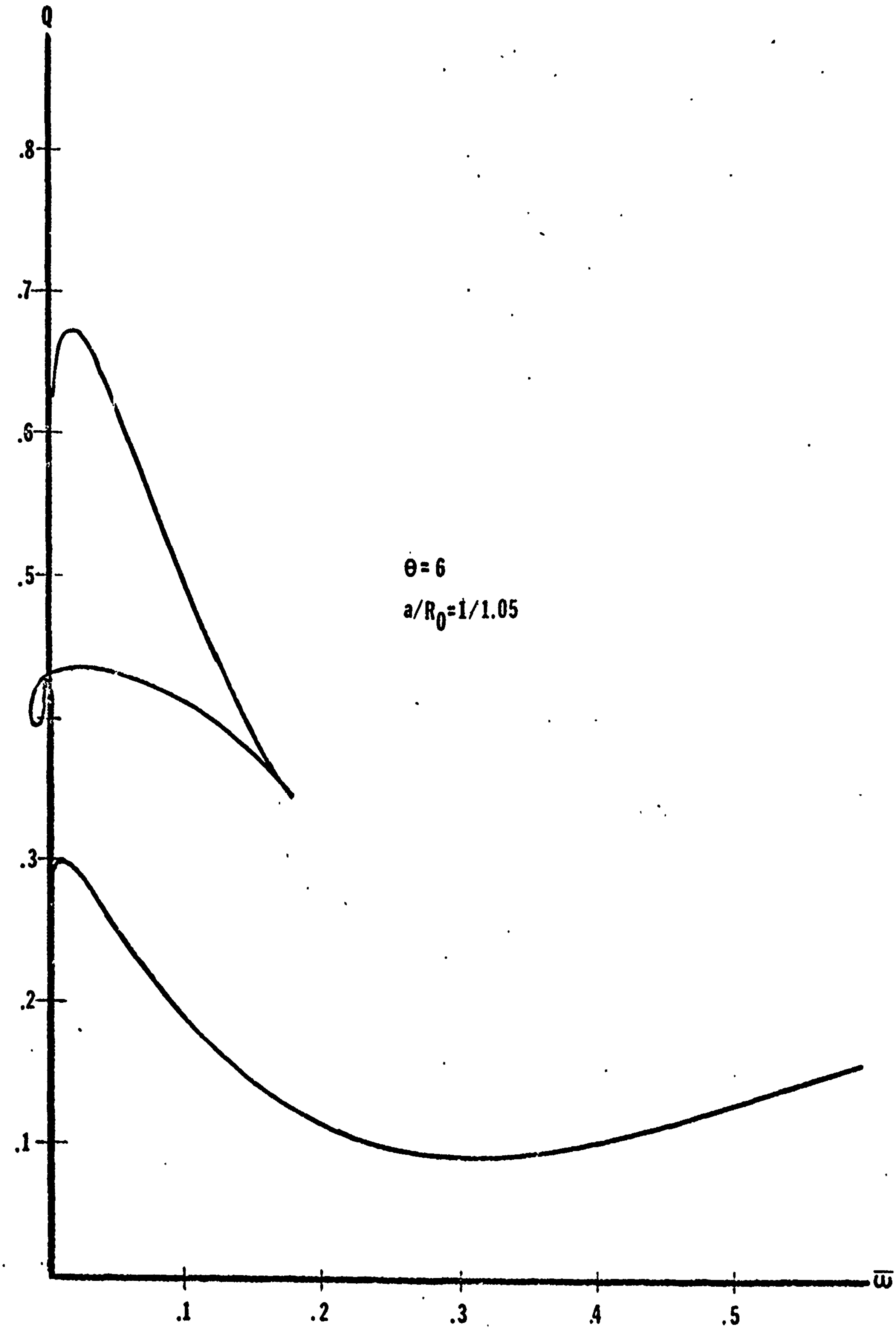

Figure 3 Load-Deflection Curve Including Higher Mode Loop

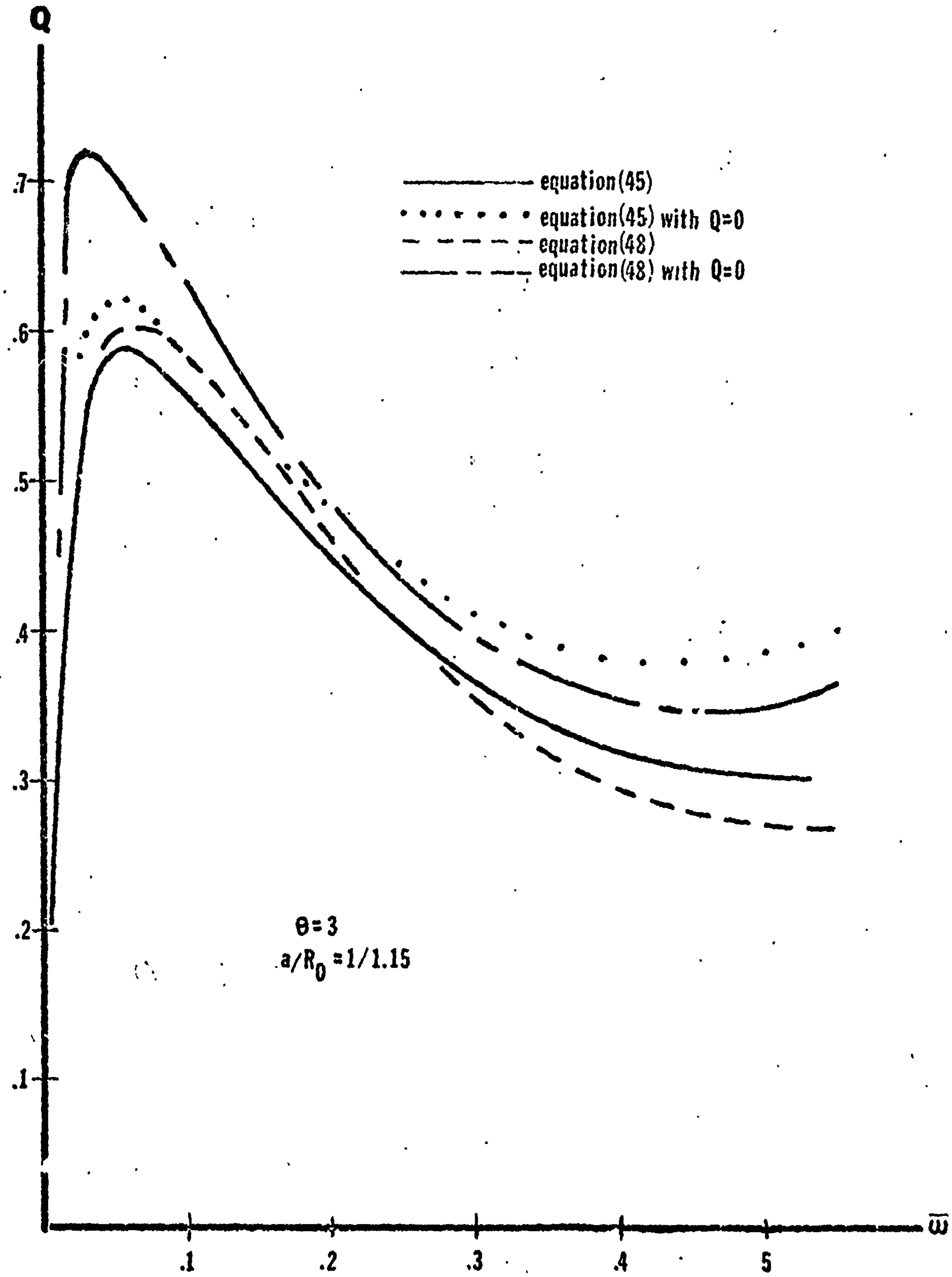

Figure 4. Effect of stiffness coefficients on load-deflection behavior

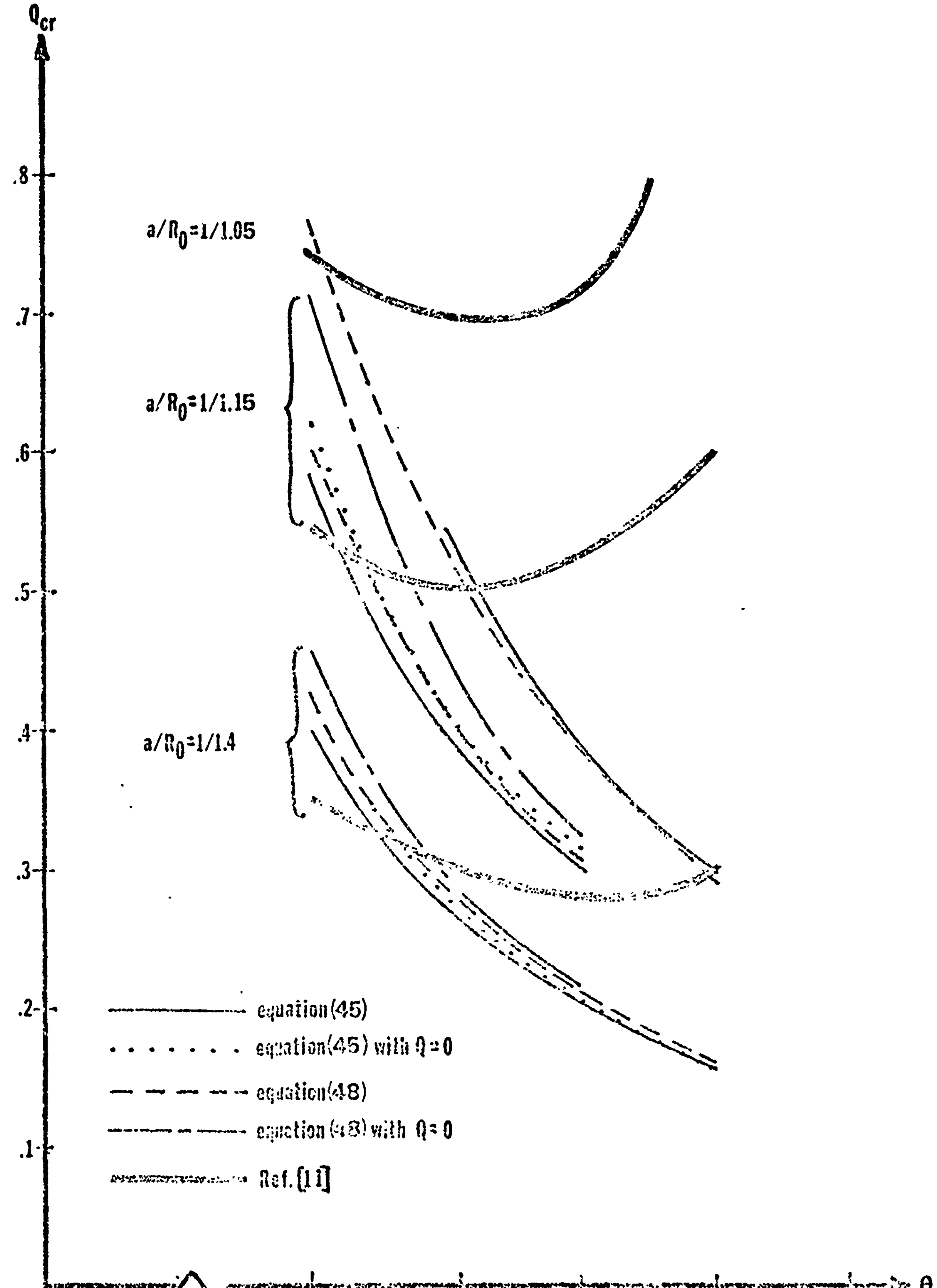

Figure 5. Collapse Pressure vs. Geometry